ROTATIONAL LINEAR WEINGARTEN SURFACES INTO THE EUCLIDEAN SPHERE

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Abstract. The aim of this paper is to present a complete description of all rotational linear Weingarten surface into the Euclidean sphere $\mathbb{S}^3$. These surfaces are characterized by a linear relation $aH+bK=c$, where H and K stand for their mean and Gaussian curvatures, respectively, whereas a, b and c are real constants.

Key words : Rotational surfaces, Linear Weingarten surfaces.

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1. Introduction and statement of results

The study of Weingarten's surface M^2 into the Euclidean space $\mathbb{R}^3$ remount to classical works development around the middle of nineteenth century by Weingarten contained in the papers [9] and [10]. Essentially these surfaces are a natural generalization of one with constant curvature, more precisely, they satisfy a relation $W(k_1, k_2) = 0$, where k_1 and k_2 stand for the principal curvatures of the surface while W is a smooth function defined over the Euclidean space $\mathbb{R}^2$, distinguishing when $W(k_1, k_2) = f(H^2 - K)$, where H and K denote, respectively, the mean and the Gaussian curvatures of M^2. We point out that replacing the Euclidean space $\mathbb{R}^3$ either by the Euclidean sphere $\mathbb{S}^3$ or by the hyperbolic space $\mathbb{H}^3(-1)$ we have the same definition. In the later case the work due to Bryant [2] when $f(H^2 - K) = c$, for a constant c, retake this subject after a long delay, as well as works due to Rosenberg and Sa Earp [8]. When the ambient space is the Euclidean sphere $\mathbb{S}^3$ the case of rotational surfaces was described by Dajczer and do Carmo [3] only for constant mean curvature. In recent works Almeida et al. [1] and Li et al. [4] obtained some results for Weingarten surfaces of the Euclidean three sphere. For the special case $U(H, K) = 0$, where U is an affine function, Lopez [5] described such surfaces into the Euclidean space $\mathbb{R}^3$ with an additional requirement on the discriminant of U. Our purpose here is to extend this later description for a class of rotational surfaces into the Euclidean sphere $\mathbb{S}^3$. Indeed, we shall give a special attention for such surfaces satisfying $U(H, K) = 0$, where the function satisfies

$$U(H, K) = aH + bK - c, \tag{1}$$

being $a, b, c \in \mathbb{R}$. Let us call such class of surfaces as Rotational Linear Weingarten Surfaces or shortly by RLWS.

We can assume, without loss of generality, that $c \geq 0$. Moreover, we choose $a \neq 0$ and $b \neq 0$, since the cases $a = 0$ and $b = 0$ were analyzed by Palmas in [6] and [7]. One fundamental ingredient to understand the behavior of a RLWS as well as its qualitative properties is the sign of the its discriminant which is defined according to $\Delta = a^2 + 4bc$. In the quoted paper López [5] described RLWS of hyperbolic type ($\Delta < 0$) in the Euclidean space $\mathbb{R}^3$ under a suitable assumption.

Following Dajczer and do Carmo [3] we shall use the terminology of rotational surface into $\mathbb{S}^3$ as a surface invariant by the orthogonal group $O(2)$ consider as a subgroup of the isometries group of $\mathbb{S}^3$. Hence we can consider a profile curve γ to describe the desired surface. Initially let us parametrize the profile curve γ in $\mathbb{S}^2$ by $\gamma(s) = \big(x(s), y(s), z(s)\big)$, with $x(s) \geq 0$. If we choose $\varphi(t) = (\cos t, \sin t)$ as an element in $O(2)$ the rotational surface generated by γ is parametrized as follows

$$\psi : M^2 \hookrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$$
$$(s,t) \mapsto (x(s)\cos t, x(s)\sin t, y(s), z(s)).$$

Moreover, we can choose the parameter s to be the arc length of γ. Then using this parameter we obtain

$$x^2(s) + y^2(s) + z^2(s) = 1\ , \ \ \dot{x}^2(s) + \dot{y}^2(s) + \dot{z}^2(s) = 1.$$

In order to compute the principal curvatures of a rotational surface $M^2 \subset \mathbb{S}^3$ we remember a fundamental lemma due to Dajczer and do Carmo [3].

Lemma 1 (Dajczer-do Carmo)**.** *Let M^2 be a rotational surface of $\mathbb{S}^3$ under the above choices. Then its principal curvatures k_1 and k_2 are given by*

$$k_1 = -\frac{\sqrt{1 - x^2 - \dot{x}^2}}{x} \ \ \textit{and} \ \ k_2 = \frac{\ddot{x} + x}{\sqrt{1 - x^2 - \dot{x}^2}}.$$

With this setting we present the fundamental relation which characterizes a RLWS in the Euclidean sphere $\mathbb{S}^3$:

$$\frac{a}{2}x\sqrt{1 - x^2 - \dot{x}^2} + \frac{b}{2}(x^2 + \dot{x}^2) + \frac{c}{2}x^2 = \alpha, \tag{2}$$

where α is a constant.

Let us denote by M_α the RLWS associated with the function x, solution of the equation (2) and the parameter α. Moreover, let us consider the special value $\alpha_0 = \dfrac{\sqrt{a^2 + (b+c)^2}}{4} + \dfrac{b+c}{4}$.

Theorem 1. *Let M_α be a RLWS with $a > 0$ and $\Delta \neq 0$. Then we have:*

1. $\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]$*;*
2. *There are no complete immersed RLWS $M_\alpha \subset \mathbb{S}^3$ that such*
$$\alpha \in \left(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\right) \cup \left(\frac{b}{2}, \frac{b+c}{2}\right);$$
3. *For any $\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)$, M_α is a complete immersed RLWS in $\mathbb{S}^3$;*
4. *There is only one complete immersed RLWS (Clifford torus) in $\mathbb{S}^3$ that such $\alpha = \alpha_0$.*

Finally we prove the the following result.

Theorem 2. *There is a family of complete immersed RLWS in $\mathbb{S}^3$ that does not contain isoparametric surfaces.*

2. Preliminaries and basic results

From now on we shall choose the discriminant $\Delta \neq 0$ and $a > 0$. An analogous analysis can be made for the case $a < 0$. First of all we begin this section by proving a lemma that establishes the fundamental relation (2).

Lemma 2. *A surface $M^2 \subset \mathbb{S}^3$ is RLWS if, and only if, the function x satisfies the following differential equation:*

$$\frac{a}{2}x\sqrt{1-x^2-\dot{x}^2}+\frac{b}{2}(x^2+\dot{x}^2)+\frac{c}{2}x^2=\alpha,$$

where α is a constant.

Proof. Taking into account that $aH + bK = c$ we use Lemma 1 to arrive at

$$\frac{a}{2}\left(\frac{\ddot{x}+x}{\sqrt{1-x^2-\dot{x}^2}}-\frac{\sqrt{1-x^2-\dot{x}^2}}{x}\right)-b\cdot\frac{\ddot{x}+x}{x}=c. \tag{3}$$

Now, note that

$$-\frac{d}{ds}\left(\frac{a}{2}x\sqrt{1-x^2-\dot{x}^2}+\frac{b}{2}(x^2+\dot{x}^2)\right)=$$

$$x\dot{x}\left[\frac{a}{2}\left(\frac{\ddot{x}+x}{\sqrt{1-x^2-\dot{x}^2}}-\frac{\sqrt{1-x^2-\dot{x}^2}}{x}\right)-b\cdot\frac{\ddot{x}+x}{x}\right].$$

Therefore, the function x satisfies the equation (3) if, and only if,

$$\frac{a}{2}x\sqrt{1-x^2-\dot{x}^2}+\frac{b}{2}(x^2+\dot{x}^2)+\frac{c}{2}x^2=\alpha,$$

where $\alpha \in \mathbb{R}$ finishing the proof of the lemma. □

Definition 1. *A solution of (2) is complete if either x is defined for all $s \in \mathbb{R}$ or if the pair $(x, \dot{x})$ admits only $(0, \pm 1)$ as limit values.*

When $(x, \dot{x})$ has $(0, 1)$ or $(0, -1)$ as limit value, we deduce that the profile curve meets orthogonally the axis of rotation. Therefore, complete solutions of the equation (2) give rise to a complete RLWS.

In order to describe the behavior of a solution of equation (2) we follow the techniques contained in the next paper [6] due to Palmas. Initially we note that a local solution x of the equation (2) paired with its first derivative $(x, \dot{x})$, is contained on a level curve of the function $F : D \to \mathbb{R}$ defined by

$$F(u,v)=\frac{a}{2}u\sqrt{1-u^2-v^2}+\frac{b}{2}(u^2+v^2)+\frac{c}{2}u^2,$$

where $D = \{(u, v) \in \mathbb{R}^2 : u \geq 0 \text{ and } u^2 + v^2 \leq 1\}$.

Lemma 3. *Let $\mathcal{P} := \{(u, v) \in int(D) : \frac{\partial F}{\partial u}(u, v) = \frac{\partial F}{\partial v}(u, v) = 0\}$ be the set of critical points of F contained in the interior of D. Then we have:*

(i) $\mathcal{P} = \{(u_+, 0)\} \Leftrightarrow b + c \geq 0$;
(ii) $\mathcal{P} = \{(u_-, 0)\} \Leftrightarrow b + c \leq 0$,

where $u_\pm^2 = \frac{1}{2}\left(1 \pm \sqrt{\frac{(b+c)^2}{a^2+(b+c)^2}}\right)$.

Proof. Straightforward calculations yield

$$\begin{aligned}
\frac{\partial F}{\partial u} &= \frac{a}{2}\sqrt{1-u^2-v^2} - a\frac{u^2}{2\sqrt{1-u^2-v^2}} + (b+c)u;\\
\frac{\partial F}{\partial v} &= -a\frac{uv}{2\sqrt{1-u^2-v^2}} + bv = \left(-a\frac{u}{2\sqrt{1-u^2-v^2}} + b\right)v.
\end{aligned}$$

For $(u,v) \in \mathcal{P}$ we affirm that $-a\dfrac{u}{2\sqrt{1-u^2-v^2}} + b \neq 0$. Otherwise from $\dfrac{\partial F}{\partial u} = 0$ we have

$$\frac{a}{2}\sqrt{1-u^2-v^2} + cu = 0.$$

Hence we conclude that $(a^2+4bc)u = \Delta \cdot u = 0$. Since $\Delta \neq 0$ and $(u,v) \in int(D)$ we arrive at a contradiction. Therefore, $v = 0$ and

$$\frac{a}{2}\sqrt{1-u^2} - \frac{a}{2}\frac{u^2}{\sqrt{1-u^2}} + (b+c)u = 0.$$

This is equivalent to

$$a(1-2u^2) = -2(b+c)\sqrt{1-u^2}. \tag{4}$$

Moreover, the solutions of the equation (4) are also solutions of the equation below

$$u^4 - u^2 + \frac{a^2}{4[a^2+(b+c)^2]} = 0. \tag{5}$$

The solutions of equation (5) are $u_{\pm}^2 = \frac{1}{2}\Big(1 \pm \sqrt{\frac{(b+c)^2}{a^2+(b+c)^2}}\Big)$. Taking into account that $\frac{1}{u_+} \cdot \frac{\partial F}{\partial u}(u_+,0) = (b+c) - |b+c|$ and $\frac{1}{u_-} \cdot \frac{\partial F}{\partial u}(u_-,0) = (b+c) + |b+c|$, we conclude

- $\frac{\partial F}{\partial u}(u_+,0) = 0 \Leftrightarrow b+c \geq 0$;
- $\frac{\partial F}{\partial u}(u_-,0) = 0 \Leftrightarrow b+c \leq 0$.

This completes the proof of the lemma. □

In what follows, let us denote by $C_\alpha = \{(u,v) \in D : F(u,v) = \alpha\}$ the level curves of the function F as well as $\alpha_{\pm} := F(u_{\pm},0)$. The next lemma enables us to determine the minimum level as well as the maximum level of F.

Lemma 4. *Under the previous assumptions the following results hold:*

(i) *If* $b+c \leq 0$*, then* $\alpha \in [\frac{b}{2}, \alpha_0]$ *and* $F^{-1}(\alpha_0) = \{(u_-,0)\}$*;*
(ii) *If* $b+c \geq 0$*, then* $\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]$ *and* $F^{-1}(\alpha_0) = \{(u_+,0)\}$*.*

Proof. We start analyzing the function F on the sets $X = D \cap \{u = 0\}$ and $Y = D \cap \mathbb{S}^1$. On the former case we have $F(u,v) = \frac{b}{2}v^2$ while on the later one $F(u,v) = \frac{b}{2} + \frac{c}{2}u^2$. Now, if $b+c \leq 0$ we get $b < 0 \leq c$, then $\min\limits_{\partial D} F = \dfrac{b}{2}$ and $\max\limits_{\partial D} F = 0 < \alpha_- = \alpha_0$. Therefore $\min\limits_{D} F = \dfrac{b}{2}$, $\max\limits_{D} F = \alpha_0$ and $F^{-1}(\alpha_0) = \{(u_-,0)\}$ because $(u_-,0)$ is the only critical point of F in $int(D)$. Now, if $b+c \geq 0$ Lemma 3 yields that $(u_+,0)$ is the only critical point of F in $int(D)$. Thereby, we have two possibilities to consider:

- $b < 0 \le c$. In this case, since $\min\limits_{\partial D} F = \dfrac{b}{2}$, $\max\limits_{\partial D} F = \dfrac{b+c}{2} < \alpha_+ = \alpha_0$, we get $\min\limits_{D} F = \dfrac{b}{2}$, $\max\limits_{D} F = \alpha_0$ and $F^{-1}(\alpha_0) = \{(u_+, 0)\}$.
- $b > 0$ and $c \ge 0$. It is easy to see that, $\min\limits_{\partial D} F = 0$ and $\max\limits_{\partial D} F = \dfrac{b+c}{2} < \alpha_+ = \alpha_0$. Therefore $\min\limits_{D} F = 0$, $\max\limits_{D} F = \alpha_0$ and $F^{-1}(\alpha_0) = \{(u_+, 0)\}$.

Thus we conclude the proof of the lemma. □

Lemma 5. *The partial derivative* $\dfrac{\partial F}{\partial u}$ *vanishes on the set*

$$\Gamma = \{(u, v) \in int(D) : 1 - u^2 - v^2 = \frac{\tau^2}{a^2} u^2\}, \tag{6}$$

where $\tau = \sqrt{a^2 + (b+c)^2} - (b+c)$.

Proof. From the expression of the partial derivatives found in the proof the Lemma 3 we deduce that $\dfrac{\partial F}{\partial u} = 0$ if, and only if,

$$\frac{a}{2}\sqrt{1 - u^2 - v^2} - a\frac{u^2}{2\sqrt{1 - u^2 - v^2}} + (b+c)u = 0. \tag{7}$$

We can suppose that $u \neq 0$, since $(u, v) \in int(D)$. Then (u, v) satisfies the relation (7) if, and only if,

$$at^2 + 2(b+c)t - a = 0, \tag{8}$$

where $t = \dfrac{\sqrt{1 - u^2 - v^2}}{u}$. Since its roots are $t_\pm = \frac{-(b+c) \pm \sqrt{a^2 + (b+c)^2}}{a}$ and $t_- < 0$ we deduce $t_+ = \dfrac{\sqrt{1 - u^2 - v^2}}{u} = \dfrac{\tau}{a}$ which is equivalent to $(u, v) \in \Gamma$. □

Geometrically, the points of the curve Γ are the points where the level curves have tangent vector parallel to the axis u.

Remark 1. *Analyzing the cases* $b + c \le 0$ *and* $b + c \ge 0$ *we conclude that:* $b + c \le 0 \Rightarrow (u_-, 0) \in \Gamma$ *whereas* $b + c \ge 0 \Rightarrow (u_+, 0) \in \Gamma$.

Lemma 6. *Under the previous notations the items below are valid.*

(i) $C_\alpha \cap \Gamma \neq \varnothing \Leftrightarrow \frac{b}{2} < \alpha \le \alpha_0$. *Moreover, if* $\alpha \in (\frac{b}{2}, \alpha_0)$ *then* $C_\alpha \cap \Gamma$ *has only two elements;*
(ii) $(u, v) \in C_\alpha \cap \{u = 0\} \Leftrightarrow b \cdot v^2 = 2\alpha$;
(iii) $(u, v) \in C_\alpha \cap \mathbb{S}^1 \Leftrightarrow c \cdot u^2 = 2\alpha - b$.

Proof. By Lemma 5 it follows that, $(u, v) \in C_\alpha \cap \Gamma$ if, and only if,

$$\frac{a}{2}u\sqrt{\frac{\tau^2}{a^2}u^2} + \frac{b}{2}\left(1 - \frac{\tau^2}{a^2}\right)u^2 + \frac{c}{2}u^2 = \alpha \Leftrightarrow$$

$$\left(\tau - \frac{b}{a^2}\tau^2 + c\right)u^2 = 2\alpha - b. \tag{9}$$

Since $\tau = \frac{a^2}{\sqrt{a^2 + (b+c)^2} + (b+c)}$ we deduce $\tau > \frac{b}{a^2}\tau^2$ otherwise

$$b\tau < \tau\big(\sqrt{a^2 + (b+c)^2} + (b+c)\big) = a^2 \le b\tau.$$

Then $(u, v) \in C_\alpha \cap \Gamma$ if, and only if, $2\alpha > b$ which yields the first item. While the second one is an immediate consequence of the equality $F(0, v) = \alpha$. Now observe that $(u, v) \in C_\alpha \cap\ \mathbb{S}^1$ if, and only if, $u^2 + v^2 = 1$ and $F(u, v) = \alpha$. Using the function F we conclude the item (iii). $\square$

3. Main result

Next we characterize the level curves of the function F.

Proposition 1. (Level Curves) *The level curves C_α of the function F satisfy:*

(1) *If $\alpha \in \big(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\big)$, then C_α intersects $\{(0, v) : -1 < v < 1\}$ at two different points. Moreover, $C_{\frac{b}{2}} \cap \{u = 0\} = \{(0, \pm 1)\}$, $C_0 \cap \{u = 0\} = \{(0, 0)\}$, $b > 0$ implies $C_0 = \{(0, 0)\}$ and $b < 0$ implies $C_{\frac{b}{2}} = \{(0, \pm 1)\}$;*

(2) *If $\big(\frac{b}{2}, \frac{b+c}{2}\big)$, then the level curve C_α intersects $\mathbb{S}^1_+ = \{(u, v) : u^2 + v^2 = 1$ and $u \geq 0\} \smallsetminus \{(0, \pm 1)\}$ at two different points. Moreover, $c = 0$ implies $C_{\frac{b}{2}} = \mathbb{S}^1_+$ and $c \neq 0$ implies $C_{\frac{b}{2}} \cap \mathbb{S}^1_+ = \{(0, \pm 1)\}$ and $C_{\frac{b+c}{2}} \cap \mathbb{S}^1_+ = \{(1, 0)\}$;*

(3) *For any $\alpha \in \big(\max\{0, \frac{b+c}{2}\}, \alpha_0\big)$, we get $C_\alpha \cap \{u = 0\} = \varnothing$ and $C_\alpha \cap\ \mathbb{S}^1_+ = \varnothing$;*

(4) *If $|b + c| = \pm(b + c)$, then $C_{\alpha_0} = \{(u_\pm, 0)\}$.*

Proof. We note that items $1, 2$ and 3 are a direct consequence of item 2 and 3 of Lemma 6. The item (4) follows directly from Lemma 4, which completes the proof of the proposition. $\square$

Corollary 1. *Under the previous assumptions the following results hold:*

(1) *If $\alpha \in \big(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\big) \cup \big(\frac{b}{2}, \frac{b+c}{2}\big)$, then the level curve C_α is not complete;*

(2) *If $\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)$, then C_α is a smooth, simple closed curve.*

Proof. If $\alpha \in \big(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\big) \cup \big(\frac{b}{2}, \frac{b+c}{2}\big)$, we get by Proposition 1 that the level curve C_α is not defined for all $s \in \mathbb{R}$. Therefore, C_α is not complete. This proof the item (1). Follows directly from Proposition 1 [item (3)] that C_α is a smooth, simple closed curve. $\square$

Proof of the Theorem 1. We follow the numbering is accordance with the statements of the theorem.

1. Follows directly from Lemma 4 that $\alpha \in [\min\{0, \frac{b}{2}\}, \alpha_0]$;
2. If the function x satisfies $F(x, \dot{x}) = \alpha$ and $\alpha \in \big(\min\{0, \frac{b}{2}\}, \max\{0, \frac{b}{2}\}\big) \cup \big(\frac{b}{2}, \frac{b+c}{2}\big)$, we get by Corollary 1 that x is not defined for all $s \in \mathbb{R}$. Therefore, the RLWS associated is not complete;
3. Next we note that item 2 of Corollary 1 yield: if $F(x, \dot{x}) = \alpha$ and $\alpha \in (\max\{0, \frac{b+c}{2}\}, \alpha_0)$ then x is defined for all $s \in \mathbb{R}$. Thereby, the RLWS associated is complete;
4. If x is such that $F(x, \dot{x}) = \alpha_0$, then $\dot{x} = 0$ and $x = u_\pm$. Therefore, the RLWS associated is a Clifford torus,

which completes the proof of the desired theorem. $\square$

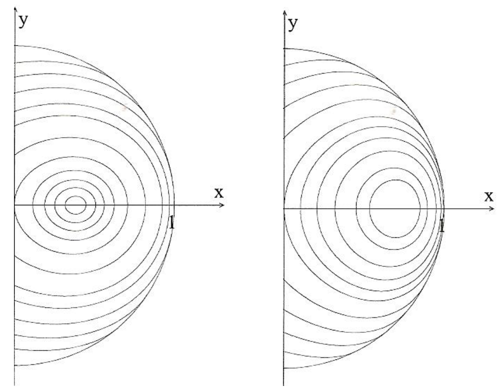

FIGURE 1. $b + c < 0$ and $b + c = 0$, respectively.

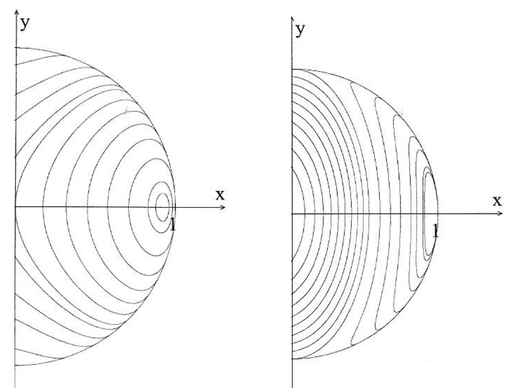

FIGURE 2. $b + c > 0$: $b < 0$ and $b > 0$, respectively.

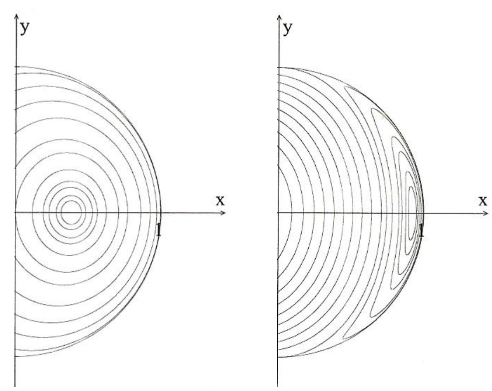

FIGURE 3. $c = 0$: $b < 0$ and $b > 0$, respectively.

In order to prove Theorem 2 we shall need the following lemma.

Lemma 7. *Let x be the solution of equation (2) such that $x(s) \neq 0$ and $\dot{x}(s) \neq 0$, $\forall\, s \in \mathbb{R}$. If $c = 0$ and k_1 is constant, then $\alpha = \frac{b}{2}$.*

Proof. By Lemma 1 we get $-k_1 x = \sqrt{1 - x^2 - \dot{x}^2}$ and $-k_1 k_2 x = x + \ddot{x}$. Next we note that if x is a solution of equation (2) and $F(x, \dot{x}) = \alpha$, then

$$ax\sqrt{1 - x^2 - \dot{x}^2} + b(x^2 + \dot{x}^2) = 2\alpha.$$

If $k_1 = 0$, we have that $x^2 + \dot{x}^2 = 1$. It follows that $F(x, \dot{x}) = \frac{b}{2}$. Now suppose $k_1 \neq 0$. In this case, $-ak_1 x^2 + b(x^2 + \dot{x}^2) = 2\alpha$. Differentiating this equality we obtain

$$-2ak_1 x\dot{x} + 2b(x + \ddot{x})\dot{x} = 0 \Leftrightarrow -2ak_1 x + 2b(x + \ddot{x}) = 0$$
$$\Leftrightarrow -a + b\frac{x + \ddot{x}}{k_1 x} = 0 \Leftrightarrow -a - bk_2 = 0 \Leftrightarrow k_2 = -\frac{a}{b}.$$

It follows from the expression of k_2 that $\sqrt{1 - x^2 - \dot{x}^2} = \frac{a}{b}x + \beta$, where $\beta \in \mathbb{R}$. Thus, $k_1 = -\frac{a}{b} - \frac{\beta}{x}$. As k_1 is constant, we deduce that $\beta = 0$ as well as $k_1 = k_2 = -\frac{a}{b}$. Therefore,

$$\begin{aligned} F(x, \dot{x}) &= \frac{a}{2}x\sqrt{1 - x^2 - \dot{x}^2} + \frac{b}{2}(x^2 + \dot{x}^2) \\ &= -\frac{a}{2}k_1 x^2 + \frac{b}{2}(1 - k_1^2 x^2) = \frac{b}{2}, \end{aligned}$$

which finishes the proof of lemma. □

Finally we shall prove the Theorem 2.

Proof of the Theorem 2. If x is solution of equation (2) such that $F(x, \dot{x}) = \alpha$ and $\alpha \in (\max\{0, \frac{b}{2}\}, \alpha_0)$, it follows from Proposition 1 that $(x, \dot{x})$ is a smooth, simple closed curve and $x(s) \neq 0\ \forall\ s \in \mathbb{R}$. Thereby, Lemma 6 enables us to suppose, without loss of generality, that $\dot{x}(s) \neq 0\ \forall\ s \in \mathbb{R}$. Therefore, it follows from Lemma 7 that when $c = 0$ the RLWS associated with x is not isoparametric. Moreover, by Theorem 1 we deduce that such surfaces are complete and immersed. This completes the proof of the theorem. □

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